La mer 24: 59-62, 1986
Société franco-japonaise d'océanographie, Tokyo

Comment on Boussinesq's long wave equation*

Motoyasu MIYATA**

Abstract: Two-dimensional non-linear motion in shallow water on a flat bottom is discussed. It is found that the non-linear equations, with the wave amplitude being not necessarily small, yield a steady solution of solitary type whose wave form is nearly equal to that obtained from the Boussinesq equations.

1. Introduction

Consider two-dimensional motion in a perfect fluid on a flat bottom. Let (x, z) be a system of Cartesian co-ordinates, and let z be measured vertically upwards from the undisturbed surface. The linear shallow water equation is (see, e.g., STOKER 1957)

$$\frac{\partial u}{\partial t}+g\frac{\partial \zeta}{\partial t}=0, \qquad (1)$$

$$\frac{\partial \zeta}{\partial t}+h\frac{\partial u}{\partial t}=0, \qquad (2)$$

where u is the horizontal velocity in x-direction, ζ the displacement of the water surface, h the undisturbed water depth, g the gravitational acceleration. In deriving Eq.'s (1) and (2), the following two length ratios were assumed to be negligible compared with unity.

$$\varepsilon\equiv\frac{A}{h}, \qquad \delta\equiv\frac{h}{L}, \qquad (3)$$

where L is the horizontal scale of the motion and A is the amplitude. The two parameters ε and δ are independent of each other and when the theory extends to include non-linear terms, the ratio of ε to δ^2 is believed to play a central role in deciding the type of the approximate solution of the full original equations. The dependence of the non-linear shallow water wave equation on the fundamental ratio was systematically analysed by URSELL (1953) so that it is often called Ursell's parameter, although STOKES (1849) was the first to call attention to it in explaining what Ursell later referred to as "long wave paradox". Ursell concluded that BOUSSINESQ's theory (1877) or the KdV equation (KORTEVEG and DE VRIES, 1895) is based on the approximation

$$\varepsilon\sim\delta^2\ll 1, \qquad (4)$$

and steady non-linear wave solution is possible in this case. He discussed that if

$$\varepsilon\gg\delta^2\ll 1, \qquad (5)$$

then non-linear effect would overcome and wave would break. Since then, in the literature dealing with the non-linear long waves, it has been commonly the custom to use this ratio to distinguish non-breaking wave from breaking wave. The present author has found it difficult to reconcile himself to these discussions, and in this paper he is going to show that Boussinesq's approximation is not necessarily the only case that produces a steady wave solution.

2. Derivation of approximate equations

The formalism of MEI (1983, Chap. 11) will be used. The variables are normalized by the following replacement.

$$x\rightarrow Lx, \quad z\rightarrow hz, \quad \zeta\rightarrow A\zeta$$

$$t\rightarrow\frac{L}{\sqrt{gh}}t, \quad u\rightarrow A\sqrt{\frac{g}{h}}u, \quad w\rightarrow\frac{AL}{h}\sqrt{\frac{g}{h}}w.$$

Now the non-dimensional Laplace equation,

$$\delta^2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0, \qquad (6)$$

must be solved for the velocity potential ϕ, subject to the boundary conditions at the free

* Received December 15, 1985

** Geophysical Institute, Faculty of Science, University of Tokyo, Hongo, Tokyo, 113 Japan

surface $z=\varepsilon\zeta$

$$\delta^2\left(\frac{\partial\zeta}{\partial t}+\varepsilon\frac{\partial\phi}{\partial t}\frac{\partial\zeta}{\partial x}\right)=\frac{\partial\phi}{\partial z}, \tag{7}$$

$$\delta^2\left(\frac{\partial\phi}{\partial t}+\zeta\right)+\frac{1}{2}\varepsilon\left\{\delta^2\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}=0, \tag{8}$$

and at the bottom $z=-1$,

$$\frac{\partial\phi}{\partial z}=0. \tag{9}$$

In terms of complex variables,

$$q=x+iz,\quad Q=\phi+i\psi,\quad \frac{dQ}{dq}=u-iw, \tag{10}$$

where ψ is the stream function. Since Q should be an analytic function which satisfies the boundary conditions (7), (8) and (9), we can formally write

$$\frac{dQ}{dq}=e^{izD}u_0,\quad u_0=\left.\frac{\partial\phi}{\partial x}\right|_{z=-1}, \tag{11}$$

where D is an operator

$$e^{izD}=1+i\delta(z+1)\frac{d}{dx}+\frac{1}{2}(i\delta)^2(z+1)^2\frac{d^2}{dx^2}+\frac{1}{3}(i\delta)^3(z+1)^3\frac{d^3}{dx^3}+\cdots\cdots\cdots.$$

Substituting (10) and (11) into (7) and (8), and ignoring the terms of order δ^4 or higher, we obtain, after some manipulation,

$$\frac{1}{3}\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial x}\left(u_0-\frac{\delta^2}{2}\frac{\partial^2u_0}{\partial x^2}\right)+\eta\frac{\partial u_0}{\partial x}-\frac{\delta^2}{6}\eta^3\frac{\partial^3u_0}{\partial x^3}=0, \tag{12}$$

$$\frac{\partial u_0}{\partial t}+\varepsilon u_0\frac{\partial u_0}{\partial x}+\frac{1}{3}\frac{\partial\eta}{\partial x}-\frac{\delta^2}{2}\frac{\partial}{\partial x}\left(\eta^2\frac{\partial^2u_0}{\partial x\partial t}\right)+\varepsilon\delta^2\frac{\partial}{\partial x}\left\{\frac{1}{2}\eta^2\left(\frac{\partial u_0}{\partial x}\right)^2-\frac{1}{3}\eta^2u_0\frac{\partial^2u_0}{\partial x^2}\right\}=0, \tag{13}$$

where η is the total depth

$$\eta=1+\varepsilon\zeta. \tag{14}$$

It should be noted that in deriving the approximate equations (12) and (13), δ is assumed to be small but ε is left arbitrary. If the depth-averaged horizontal velocity U is defined by

$$U=\int_{-1}^{\varepsilon\eta}udz=u_0-\frac{\delta^2}{6}\eta^2\frac{\partial^2u_0}{\partial x^2}+0(\delta^4), \tag{15}$$

which is inverted to yield

$$u_0=U+\frac{\delta^2}{6}\eta^2\frac{\partial^2U}{\partial x^2}+0(\delta^4), \tag{16}$$

then Eq.'s (12) and (13) can be rewritten as

$$\frac{\partial\eta}{\partial t}+\varepsilon\frac{\partial}{\partial x}(\eta U)=0, \tag{17}$$

$$\frac{\partial U}{\partial t}+\varepsilon U\frac{\partial U}{\partial x}+\frac{1}{\varepsilon}\frac{\partial\eta}{\partial x}+\frac{\delta^2}{6}\frac{\partial}{\partial t}\left(\eta^2\frac{\partial^2U}{6x^2}\right)-\frac{\delta^2}{2}\frac{\partial}{\partial x}\left(\eta^2\frac{\partial^2U}{\partial x\partial t}\right)+\varepsilon\delta^2\frac{\partial}{\partial x}\left\{\frac{1}{2}\eta^2\left(\frac{\partial U}{\partial x}\right)^2-\frac{1}{3}\eta^2U\frac{\partial^2U}{\partial x^2}\right\}=0. \tag{18}$$

It is interesting to note that Eq. (17), which represents the depth-averaged conservation of mass, is exact to all orders of δ.

For convenience, Eq. (18) is further changed to another form by using (17). Since

$$\begin{aligned}\frac{1}{2}\eta^2\left(\frac{\partial U}{\partial x}\right)^2-\frac{1}{3}\eta^2U\frac{\partial^2U}{\partial x^2}&=\frac{1}{2}\eta^2\left(\frac{\partial U}{\partial x}\right)^2+\frac{1}{3}\left(\frac{\partial\eta}{\partial x}\right)^{-1}\left\{\eta^3\frac{\partial U}{\partial x}-\eta^2\frac{\partial}{\partial x}(\eta U)\right\}\frac{\partial^2U}{\partial x^2}\\&=\frac{1}{6}\left(\frac{\partial\eta}{\partial x}\right)^{-1}\frac{\partial}{\partial x}\left\{\eta^3\left(\frac{\partial U}{\partial x}\right)^2\right\}+\frac{1}{3\varepsilon}\left(\frac{\partial\eta}{\partial x}\right)^{-1}\eta^2\frac{\partial\eta}{\partial t}\frac{\partial^2U}{\partial x^2},\end{aligned} \tag{19}$$

Eq. (18) is equivalent to:

$$\begin{aligned}&\frac{\partial U}{\partial t}+\varepsilon U\frac{\partial U}{\partial x}+\frac{1}{\varepsilon}\frac{\partial\eta}{\partial x}+\frac{\varepsilon\delta^2}{6}\frac{\partial}{\partial x}\times\left[\left(\frac{\partial\eta}{\partial x}\right)^{1-}\frac{\partial}{\partial x}\left\{\eta^3\left(\frac{\partial U}{\partial x}\right)^2\right\}\right]\\&+\frac{\delta^2}{6}\frac{\partial}{\partial t}\left(\eta^2\frac{\partial^2U}{\partial x^2}\right)-\frac{\delta^2}{2}\frac{\partial}{\partial x}\left(\eta^2\frac{\partial^2U}{\partial x\partial t}\right)+\frac{\delta^2}{3}\frac{\partial}{\partial x}\left\{\left(\frac{\partial\eta}{\partial x}\right)^{-1}\eta^2\frac{\partial\eta}{\partial t}\frac{\partial^2U}{\partial x^2}\right\}=0.\end{aligned} \tag{20}$$

3. Steady wave solution

A steady wave solution of Eq.'s (17) and (20) will be sought. If the new variable ξ defined by

$$\xi=x-ct \tag{21}$$

is introduced, then Eq. (17) becomes

$$-c\frac{d}{d\xi}+\varepsilon\frac{d}{d\xi}(\eta U)=0. \tag{22}$$

Upon integration with respect to ξ,

$$\varepsilon U=\frac{B}{\eta}+C, \tag{23}$$

is obtained, where B is a constant.

Equation (20) is simplified to

$$-c\frac{dU}{d\xi}+\varepsilon U\frac{dU}{d\xi}+\frac{1}{\varepsilon}\frac{d\eta}{d\xi}+\frac{\varepsilon\delta^2}{6}\times\frac{d}{d\xi}\left[\left(\frac{d\eta}{d\xi}\right)^{-1}\frac{d}{d\xi}\left\{\eta^3\left(\frac{dU}{d\xi}\right)^2\right\}\right]=0, \tag{24}$$

the last three terms cancelling out one another.

Substitution of (23) in (24) leads to:

$$-\frac{B^2}{\eta^3}\frac{d\eta}{d\xi}+\frac{d\eta}{d\xi}+\frac{\delta^2B^2}{6}\times\frac{d}{d\xi}\left[\left(\frac{d\eta}{d\xi}\right)^{-1}\frac{d}{d\xi}\left\{\frac{1}{\eta}\left(\frac{d\eta}{d\xi}\right)^2\right\}\right]=0. \tag{25}$$

Integration of this equation twice yields

$$-\frac{B^2}{2\eta}+\frac{1}{2}\eta^2+\frac{\delta^2B^2}{6\eta}\left(\frac{d\eta}{d\xi}\right)^2=D\eta+E, \tag{26}$$

where both D and E are constant.

Assume that Eq. (26) has a solution of solitary type:

$$\eta\to1,\ \frac{d\eta}{d\xi},\ \frac{d^2\eta}{d\xi^2},\ U\to0\ \text{as}\ |\xi|\to\infty. \tag{27}$$

Then, from (23) and (25),

$$B=-C,\ D=1+\frac{C^2}{2},\ E=\frac{1}{2}-C^2. \tag{28}$$

Using (14) and (28), Eq. (25) can be rearranged to:

$$\delta^2c^2\left(\frac{d\zeta}{d\xi}\right)^2=3\zeta^2(c^2-1-\varepsilon\zeta), \tag{29}$$

which is integrated to yield

$$\zeta=\operatorname{sech}^2\frac{1}{2\delta}\sqrt{\frac{3\varepsilon}{1+\delta}}\,\xi \tag{30}$$

with

$$c^2=1+\varepsilon. \tag{31}$$

In dimensional form, the wave profile is

$$\zeta=A\operatorname{sech}^2\sqrt{\frac{3A}{h+A}}(x-ct) \tag{32}$$

with

$$c^2=gh\left(1+\frac{A}{h}\right).$$

Besides the solitary wave just obtained, periodic permanent waves of cnoidal type are possible, but they are not discussed here.

4. Discussions and concluding remarks

If the condition (4) is assumed, Eq.'s (17) and (18) are simplified to:

$$\frac{\partial\zeta}{\partial t}+\frac{\partial}{\partial x}\{(1+\varepsilon\zeta)U\}=0, \tag{33}$$

$$\frac{\partial U}{\partial t}+\varepsilon U\frac{\partial U}{\partial t}+\frac{\partial\zeta}{\partial t}-\frac{\delta^2}{3}\frac{\partial^3U}{\partial x^2\partial t}=0. \tag{34}$$

Equations (33) and (34) are called the Boussinesq equations. The well-known KdV equation is essentially the same as these equations. They are known to have a steady wave solution of solitary type which is, in physical variables,

$$\zeta=A\operatorname{sech}^2\frac{1}{2h}\sqrt{\frac{3A}{h}}(x-ct). \tag{35}$$

It is rather remarkable that although Eq. (18) significantly differs from (34), the solitary wave solution (32) is in form almost the same as the Boussinesq or the KdV solution (35). This implies that as far as the steady solution is concerned, the pair of the Boussinesq equations or the KdV equation happens to be a good approximation to the equations (17) and (18). However, unsteady motion is governed by different equations and the highly non-linear equation (18) [can be considered to be a large-amplitude generalization of Eq. (34).

In water of a single layer under consideration, the greatest height of solitary wave is determined by dynamical requirements. Both MCCOWAN (1894) and LENAW (1966) predicted the maximum of ε to be 0.83: STRELKOFF (1971) and FENTON (1972) obtained the value 0.85. In any case ε is usually small compared with unity. However, for internal waves the equivalent ε may be larger than unity and the equations corresponding to (17) and (18) should yield a steady solution whose wave form is quite different from that obtained from the Boussinesq or the KdV equation. MIYATA's internal solitary wave of large amplitude (1985) in a two-fluid system is an example. The time-dependent equations for non-linear internal waves in shallow water can be derived in a similar though more complicated way (MIYATA, under preparation).

It is concluded that the non-linear equations in shallow water of constant depth with ε being left arbitrary have a steady wave solution of solitary type whose wave form is almost the same as that obtained by the Boussinesq equations.

The author would like to thank Dr. H. TAKEDA for helpful discussions. Typing was done by Ms. T. OSADA.

References

BOUSSINESQ, M. J. (1872): Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. J. mathématiquées, **17**, 55-108.

FENTON, J. (1972): A nineth-order solution for the solitary wave. J. Fluid Mech., **53**, 257-271.

KORTEWEG, D. J. and G. DE VRIES (1895): On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves. Phil. Mag., **39**, 422-443.

LENAU, C. W. (1966): The solitary wave of maximum amplitude. J. Fluid Mech., **26**, 309-320.

MCCOWAN, J. (1894): On the highest wave of permanent type. Phil. Mag., **38**, 351-358.

MEI, C. C. (1983): *The Applied Dynamics of Ocean Surface Waves.* Wiley. 740 pp.

MIYATA, M. (1985): An internal solitary wave of large amplitude. La mer, **23**, 40-45.

STOKER, J. J. (1957): *Water Waves.* Interscience, New York. 567 pp.

STOKES, G. G. (1849): On the theory of oscillatory waves. Trans. Camb. Phil. Soc., **8**, 441-445 (Math. and phys. pap., **1**, 197-229, Cambridge, 1880).

STRELKOFF, T. (1971): An exact numerical solution of the solitary wave. Proc. 2nd Int. Conf. Num. Methods Fluid Dyn., Springer.

URSELL, F. (1953): The long-wave paradox in the theory of gravity waves. Proc. Camb. Phil. Soc., **49**, 685-694.

ブーシネスクの長波方程式について

宮 田 元 靖

要旨: 非線型浅水波方程式に検討を加えた。その結果ソリトン解を持つ近似方程式が得られた。これはブーシネスクの長波方程式とは多少異なっているので両者を比較して議論した。