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# Grid size and the biharmonic form for subgrid-scale diffusion in a finite difference vorticity equation in the ocean*

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The finite difference vorticity equation in the ocean can not be solved unless the grid size is smaller than a critical value depending on the coefficient of subgrid-scale diffusion. A previous paper (TAKANO, 1975) deals with the case where the conventional harmonic form is used for the subgrid-scale diffusion. Recently much more scale-selective biharmonic form is sometimes used instead of the harmonic form.

In this context, the present note is concerned with the relationship between the grid size and the coefficient of subgrid-scale diffusion in biharmonic form.

A linear, one-dimensional vorticity equation on a $\beta$-plane is given by

$$B\frac{\partial^6\phi}{\partial x^6}+\beta\frac{\partial\phi}{\partial x}+\text{curl}\,\tau=0\,, \qquad (1)$$

where the $x$-axis is directed eastward, $\phi$ is the masstransport stream function, $\tau$ is the wind stress and $B$ is the coefficient of subgrid-scale diffusion in biharmonic form.

External forcing, curl $\tau$, is of no importance for the purpose of the present study, so that it is ignored. The centered finite difference analog of (1) is

$$\phi_{n+3}-6\,\phi_{n+2}+(15+d^2/2)\,\phi_{n+1}-20\,\phi_n$$
$$+(15-d^2/2)\,\phi_{n-1}-6\,\phi_{n-2}+\phi_{n-3}=0\,, \qquad (2)$$

where $d=l\varDelta x$, $\varDelta x$ is the grid size, $l=(B/\beta)^{-1/5}$, and subscripts $(n\pm3)$, $(n\pm2)$, $(n\pm1)$ and $n$ refer to $(n\pm3)$, $(n\pm2)$, $(n\pm1)$ and $n$th grid points.

If the amplification factor $\delta$ is introduced, then $\phi_{n+1}=\delta\phi_n$. Eq. (2) becomes

$$\delta^6-6\,\delta^5+(15+d^2/2)\delta^4-20\,\delta^3$$
$$+(15-d^2/2)\delta^2-6\,\delta+1=0\,. \qquad (3)$$

The amplification factors $\delta_m\,(m=1,\ldots,6)$ derived from the differential vorticity equation $(B\partial^6\phi/\partial x^6+\beta\partial\phi/\partial x=0)$ are given by $\exp(\sigma_m d)$, where $\sigma_m$ are the roots of

$$\sigma^6+\sigma=0\,,$$

i.e.,
$$\sigma_1=0\,,$$
$$\sigma_2=-1\,,$$
$$\sigma_{3,4}=p\pm iq\,, \qquad (4)$$
$$\sigma_{5,6}=r\pm is\,,$$

where $p=\cos(3\pi/5)$, $q=\sin(3\pi/5)$, $r=\cos(\pi/5)$ and $s=\sin(\pi/5)$.

The amplification factors $\delta_m$ and $\delta_m'$ are shown for $d$ in Figs. 1 and 2. The real and imaginary parts of them are shown if they are complex numbers.

One of the amplification factors $\delta_1$ obtained from (3) is 1, which is identical with $\delta_1'$ (=exp $(\sigma_1 d)$). Both are not figured. There is no significant difference between $\delta_2$ and $\delta_2'$. The imaginary part of $\delta_5$ and $\delta_6$ is very close to that of $\delta_5'$ and $\delta_6'$, though the real part $R(\delta_{5,6})$ is fairly larger than $R(\delta_5',_6)$ for $d>2.0$.

The amplification factors $\delta_3$ and $\delta_4$ are close to each other for $d<1.5$, and not so far from each other for $1.5<d<2.57$. However, $\delta_{3,4}$ turn out to be real, negative numbers for $d\gtrsim 2.57$, while $\delta_3'$ and $\delta_4'$ are complex numbers, irrespective of $d$. The negative amplification factors give rise to false computational oscillation with a wave length of $2d$.

Therefore, the condition required of $d$ is

$$d\lesssim 2.57 \quad (\varDelta x_B\lesssim 2.57\,(B/\beta)^{1/5})\,. \qquad (5)$$

When the subgrid-scale diffusion is written as

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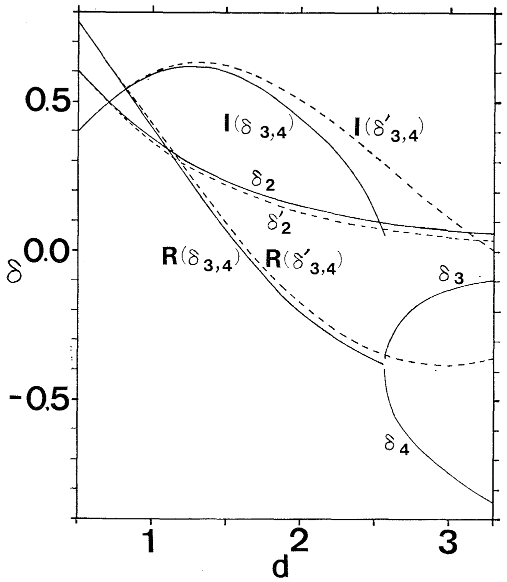


Fig. 1. Amplification factors $\delta_m$ and $\delta_m'$ ($m=2$, 3, 4) for $d$. The real and imaginary parts $R$, $R'$, $I$ and $I'$ are depicted if $\delta_m$ and $\delta_m'$ ($m=$ 3, 4) are complex numbers.

$A\partial^4\phi/\partial x^4$ instead of $-B\partial^6\phi/\partial x^6$, the required condition is

$$\Delta x_A \lesssim 2.75\,(A/\beta)^{1/3}, \qquad (6)$$

as shown in the paper cited above (TAKANO, 1975). Subscripts $_A$ and $_B$ refer to harmonic and biharmonic form.

One way of determining $B$ from $A$ is shown by SEMTNER and MINTZ (1977). If damping brought about by $-B\partial^6\phi/\partial x^6$ is made equal to that brought about by $A\partial^4\phi/\partial x^4$ at a wave length of $2\Delta x$, the shortest wavelength resolvable by the grid size $\Delta x$, then it follows that

$$B=A(\Delta x)^2/4\,, \qquad (7)$$

which gives, by use of (5),

$$\Delta x_B<3.04\,(A/\beta)^{1/3}. \qquad (8)$$

Comparison of (8) with (6) shows that for a given $A$ (or $B$) the grid size can be larger by about 10 % with biharmonic form than with harmonic form.

Next, the inequality (5) or (8) is rewritten in terms of the width of the western bounday current $W_B$.

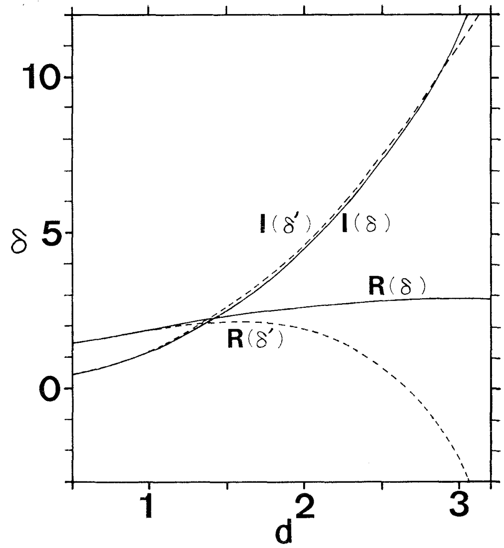


Fig. 2. Same as Fig. 1 except for $m$ ($m=5, 6$).

If the width of the ocean $L$ is much larger than $l^{-1}$, then the solution of Eq. (1) is given by

$$\begin{aligned}\phi=&-\frac{\mathrm{curl}\,\tau}{\beta}[x-L+Le^{-lx}/(2p+2)\\&-(2p+1)Le^{plx}\cos(qlx+2\pi/5)/\{2(p+1)p\}\\&-e^{rl(x-L)}\{2r\cos\{sl(x-L)\}/l-(2r^2-1)\\&\times\sin\{sl(x-L)\}/sl]\,, \end{aligned}\qquad (9)$$

with the boundary conditions

$$\phi=\frac{\partial\phi}{\partial x}=\frac{\partial^2\phi}{\partial x^2}\ \text{for } x=0 \text{ and } L.$$

The width of the western boundary current approximately calculated from the fourth term in the parentheses in (9) is $W_B \doteqdot 4.29/l$. If the 3rd term is taken into account, the width is more accurately given by

$$W_B \doteqdot 4.34/l\,, \qquad (10)$$

which leads to

$$\Delta x_B<0.592\,W_B\,. \qquad (11)$$

In the case of harmonic form, the width of the western boundary current $W_A$ is approximately

$$W_A \doteqdot 2\pi/\sqrt{3}\cdot(A/\beta)^{1/3}, \qquad (12)$$

which leads to

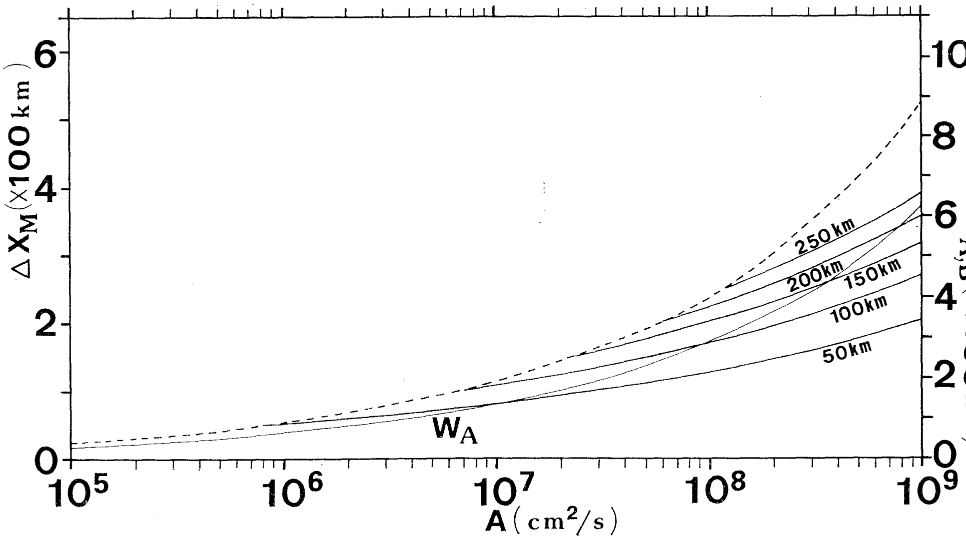


Fig. 3. Maximum grid size $\varDelta x_M$ and the width of the western boundary current $W_B$ against the coefficient of diffusion $A$ for $\varDelta x$=50, 100, 150, 200, 250 km. The maximum grid size for a specified $A$ is shown by a broken line. The width of the western boundary current $W_A$ in the case of harmonic form is also shown by a thin line.

$$\varDelta x_A < 0.76\ W_A. \qquad (13)$$

Figure 3 shows the maximum grid size $\varDelta x_M$ $(=2.57(B/\beta)^{1/5})$ as a function of $A$ and $\varDelta x$ for $\beta=2\times10^{-13}\ \mathrm{cm^{-1}\,sec^{-1}}$.

The coefficient of diffusion should be chosen in such a way that a specified grid size is less than $\varDelta x_M$. The broken line shows $\varDelta x_M$ $(=3.04\,(A/\beta)^{1/3})$, the maximum grid size for a specified $A$. The scale for the width of the western boundary current $(=\varDelta x_M/0.592)$ is also given at the right margin.

For comparison, the width of the boundary current $W_A$ is also shown by using Eq. (12).

Although the biharmonic form complicates the formulation of the boundary conditions, it is not only much more scale-selective than the conventional harmonic form but also allows a larger grid size for a specified coefficient of diffusion.

## References


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# 海の差分渦度方程式での格子間隔と重調和型うず粘性係数

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**要旨**: 海の渦度方程式を差分化して解く場合，格子間隔をうず粘性係数に依存したある値より小さくしないと解が得られない。規模選択性の強い重調和型のうず粘性を扱う。重調和型を使うと調和型粘性の場合に比べて格子間隔をやや広くとることができる。