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Relationship between the grid size and the coefficient of subgrid-scale diffusion in a finite difference advection-diffusion equation*

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A false oscillation with a wave length of two times the grid size arises from a finite difference advection- diffusion equation if the grid size is greater than a critical value determined by the coefficient of subgrid-scale diffusion.

For simplicity, a one-dimensional, steady state advection-diffusion equation is dealt with;

$$u\frac{\partial T}{\partial x}=-B\frac{\partial^4 T}{\partial x^4}, \qquad (1)$$

where u is the x-component of the velocity assumed to be positive, T is a state variable, say, temperature, and B is the coefficient of subgrid-scale diffusion. Both u and B are assumed to be constant.

Instead of the biharmonic form $-B\partial^4T/\partial x^4$, the diffusion term is conventionally written in harmonic form as $A\partial^2T/\partial x^2$.

The relationship between A and B is determined as follows (SEMTNER and MINTZ, 1977).

When centered differencing is used, both terms are approximated by

$$-B\frac{\partial^4 T}{\partial x^4}=-B \times \frac{T_{n+2}-4T_{n+1}+6T_n-4T_{n-1}+T_{n-2}}{(\Delta x)^4}, \qquad (2)$$

$$A\frac{\partial^2 T}{\partial x^2}=A\frac{T_{n+1}-2T_n+T_{n-1}}{(\Delta x)^2}, \qquad (3)$$

where Δx is the grid size, $T_{n\pm2}$, $T_{n\pm1}$ and T_n are the temperatures at the $(n\pm2)$, $(n\pm1)$ and nth grid points. With $T=qe^{ikx}$ $(i=\sqrt{-1})$, the right-hand side of (2) becomes

$$-q\frac{16B}{(\Delta x)^4}\left(\sin\frac{k\Delta x}{2}\right)^4, \qquad (4)$$

and that of (3) becomes

$$-q\frac{4A}{(\Delta x)^2}\left(\sin\frac{k\Delta x}{2}\right)^2. \qquad (5)$$

The shortest wave length resolvable by a grid size of Δx is $2\Delta x$, so that $\sin(k\Delta x/2)=1$. If the magnitude of (4) is made equal to that of (5) at this shortest wave length, then it follows that

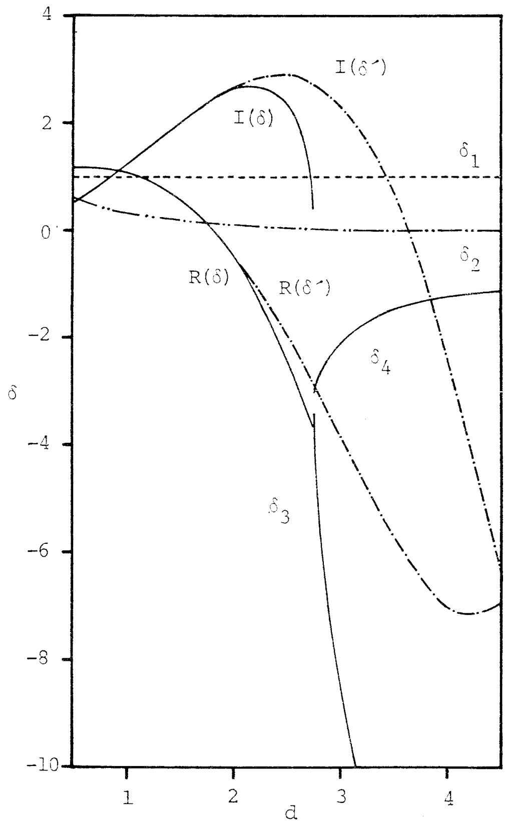

Fig. 1. Amplification factors δ_m obtained from the finite difference equation and amplification factors δ_m' obtained from the differential equation. Real and imaginary parts $R(\delta)$ and $I(\delta)$ are shown if δ_m are complex numbers.

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$$B=(\Delta x)^2 A/4 . \qquad (6)$$

When the amplification factor is denoted by δ, the centered differencing of Eq. (1) leads to

$$\delta^4+(0.5d-4)\delta^3+6\delta^2 -(0.5d+4)\delta+1=0 , \qquad (7)$$

where

$$d=(u/B)^{1/3}\Delta x . \qquad (8)$$

Figure 1 shows the four roots $\delta_m\ (m=1, .., 4)$ as functions of d. The real and imaginary parts are shown for δ_3 and δ_4 if these are complex numbers.

On the other hand, substitution of $T\propto e^{\sigma x}$ into (1) leads to

$$u\,\sigma=-B\sigma^4,$$

which gives

$$\sigma_1=0 ,$$
$$\sigma_2=-(u/B)^{1/3},$$
$$\sigma_{3,4}=(u/B)^{1/3}\frac{1\pm\sqrt{3}i}{2}.$$

The amplification factors δ_m' to be compared with δ_m are given by $\exp(\Delta x\cdot\sigma_m)\ (m=1, .., 4)$, which are also shown in Fig. 1. The real and imaginary parts are shown for δ_3' and δ_4'. Obviously δ_1 is identical with δ_1' ($\delta_1=\delta_1'=1$) irrespective of d. No significant difference is found between δ_2 and δ_2' in the practical range of d. As is readily seen however, δ_3 and δ_4 are qualitatively different from δ_3' and δ_4' for $d<2.748$, although they agree well with δ_3' and δ_4' for $d<2.0$. Both δ_3 and δ_4 are real, negative numbers for $d>2.748$, which gives rise to a false oscillation with a wave length of $2\Delta x$.

The condition necessary for getting a solution of Eq. (1) turns out to be

$$d<2.748 , \qquad (9)$$

or with (6) and (8),

$$0.193\,u\Delta x<A . \qquad (10)$$

When the diffusion term is formulated by $A\partial^2T/\partial x^2$, the condition (TAKANO, 1974) corresponding to (10) is

$$0.5\,u\Delta x<A . \qquad (11)$$

Therefore, compared with harmonic diffusion, biharmonic diffusion allows a smaller A for a given Δx, or a larger Δx for a given A. This is an advantage of $-B\partial^4T/\partial x^4$ over $A\partial^2T/\partial x^2$. As already pointed out (SEMTNER and MINTZ, 1977), the primary advantage is that the biharmonic formulation is highly scale selective because of a factor of $(\sin k\Delta x/2)^4$ in (4) in place of a factor of $(\sin k\Delta x/2)^2$ in (5); the ratio of $-B\partial^4T/\partial x^4$ to $A\partial^2T/\partial x^2$ is 0.095 for a wave length of $10\Delta x$ and 0.024 for a wave length of $20\Delta x$. Biharmonic formulation brings about very weak diffusion for wave lengths larger than $2\Delta x$.

References

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移流・拡散差分方程式での格子間隔とうず拡散係数の関係

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要旨: 移流・拡散差分方程式で重調和関数形の拡散項を用いた場合，正しい近似解を得るための拡散係数の大きさ-格子間隔の関係を調べた。これまでしばしば使われてきた調和関数形に比べて，重調和関数形には，拡散が弱いにもかかわらず，大きな格子間隔を使えるという利点がある。長波長の変化に対しては，この利点はますます大きくなる。